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An Effectiveness Prediction Model for Submarine Missile Systems

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PREFACE

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This technical report describes a real-time procedure for predicting the effectiveness of submarine missile attacks against prescribed target threats. Mathematical relationships are established which allow the probability of target damage to be expressed in terms of missile aimpoint error statistics, weapon damage characteristics, and target localization error statistics. These formulas are subsequently transformed into a compact computational algorithm, suitable for making on-line weapon performance predictions via digital computer.

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LIST OF SYMBOLS

$\underline{X}_w$	actual weapon position
$\underline{X}_t$	actual target position
$\underline{X}_{to}$	target position before alertment
$\underline{X}_{ta}$	target position at detonation
P_w	covariance matrix associated with weapon position
P_{to}	covariance matrix associated with target position before alertment
$\hat{\underline{X}}_w$	mean value of weapon position
$\hat{\underline{X}}_{ta}$	mean value of target position
R_e	radius of circumscribing circle of target escape region
R	relative distance between weapon and target at time of detonation
$p(\underline{x})$	two-dimensional probability density function
$\Phi_x(\underline{\omega})$	two-dimensional characteristic function
$L_m^{(p)}(X)$	Laguerre polynomials
J	Jacobian matrix
$J_o(\omega R)$	Zeroth order Bessel function
I	identity matrix
$\{'\}$	transpose of matrix

AN EFFECTIVENESS PREDICTION MODEL FOR SUBMARINE MISSILE SYSTEMS

I. INTRODUCTION

One measure of the effectiveness of a submarine missile attack is given by the radial distance between the target and the missile detonation point. Varying degrees of damage will be inflicted by simply placing the missile in proximity to the target at time of detonation. Accordingly, range from weapon to target is perhaps the most important single parameter affecting missile performance in any tactical situation. Under realistic operating conditions, discrepancies between the actual and estimated values of range will invariably occur because of missile aimpoint delivery errors, as well as errors inherent in the target localization process. To reliably predict weapon performance, these errors must be accurately modelled.

This report describes a real-time computational procedure for predicting the effectiveness of a submarine-launched missile attack against prescribed target threats. A functional relation is developed which expresses the probability of target damage in terms of missile aimpoint delivery error statistics, weapon damage characteristics, and target localization error statistics. This functional relation is subsequently transformed into a digital computer algorithm suitable for making on-line weapon performance predictions.

In section II the problem is formulated, and equations for the probability of target damage are developed. Section III contains approximations to the probability distributions, while section IV presents closed-form expression for probability of target damage.

II. PROBLEM FORMULATION

Consider the geometric description of a missile attack depicted in figure 1. Initially, the weapon is launched from own ship and directed to an aimpoint specified by the vector $\hat{\underline{X}}_w' = (\hat{X}_w, \hat{Y}_w)$. However, because of inherent errors in the missile delivery system, the weapon will not necessarily impact at its intended *aimpoint*. To accommodate possible discrepancies between these two points, the actual weapon position at time of detonation will be denoted by $\underline{X}_w' = (X_w, Y_w)$. For computational convenience, all errors associated with weapon positioning are tacitly assumed to be Gaussian-distributed with the following statistics:

$$E\{\underline{X}_w\} = \hat{\underline{X}}_w \tag{1a}$$

$$E\{(\underline{X}_w - \hat{\underline{X}}_w)(\underline{X}_w - \hat{\underline{X}}_w)'\} = P_w \tag{1b}$$

where the prime symbol ' is used to denote matrix transposition, and $E\{\cdot\}$ is the statistical expectation operator.

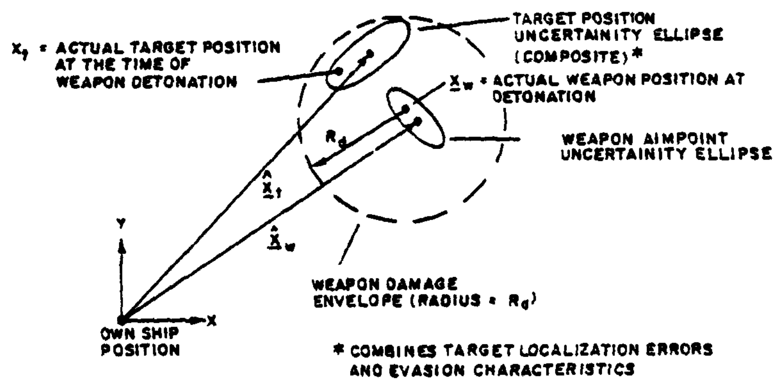


Figure 1. Geometric Description of a Missile Attack

In a similar manner, the actual and estimated values of target position at time of weapon detonation will be defined by $\underline{X}_t' = (X_t, Y_t)$ and $\hat{\underline{X}}_t' = (\hat{X}_t, \hat{Y}_t)$, respectively. To determine the statistical characteristics of $\underline{X}_t$, observe (in figure 2) that

$$\underline{X}_t = \underline{X}_{to} + \underline{X}_{ta} \tag{2}$$

where $\underline{X}_{to}$ represents target position prior to alertment of a missile attack. Estimates of the vector $\underline{X}_{to}$ are typically obtained from shipboard target motion analysis algorithms which may also provide statistical descriptions of the associated estimation errors. Since these errors are modelled as Gaussian random variables (reference 1), $\underline{X}_{to}$ will also be Gaussian-distributed with mean and covariance given by

$$E\{\underline{X}_{to}\} = \hat{\underline{X}}_{to} \tag{3a}$$

$$E\{(\underline{X}_{to} - \hat{\underline{X}}_{to})\ (X_{to} - \hat{\underline{X}}_{to})'\} = P_{to} \quad . \tag{3b}$$

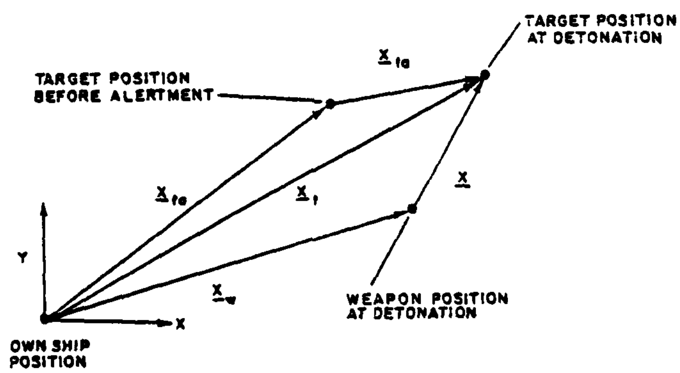


Figure 2. Vector Diagram of Relative Target Position at Time of Weapon Detonation

The vector $\underline{X}_{ta}$ defines the target position at time of weapon detonation relative to its position at time of alertment. Unfortunately, errors associated with estimating $\underline{X}_{ta}$ have a complicated non-Gaussian probability distribution that functionally depends upon the target evasion characteristics. To devise tractable approximations for this distribution requires a suitable geometric model of the target escape region. The simplest model is a circle centered at $\underline{X}_{to}$, whose radius increases with time in proportion to the maximum target speed. Although such models are frequently used in practical applications, they are somewhat unrealistic because they implicitly assume that the target has a zero turning radius. Recently, more realistic techniques of modeling and target evasion characteristics have been developed. The resulting escape regions, shown in figure 3, accurately reflect constraints on the target dynamics by utilizing a non-zero turning radius in conjunction with a finite maximum speed. At any prescribed time after alertment, the target must reside within a region whose boundary consists of an involute (dotted curve), and arcs of the two minimum-turning-radius circles.

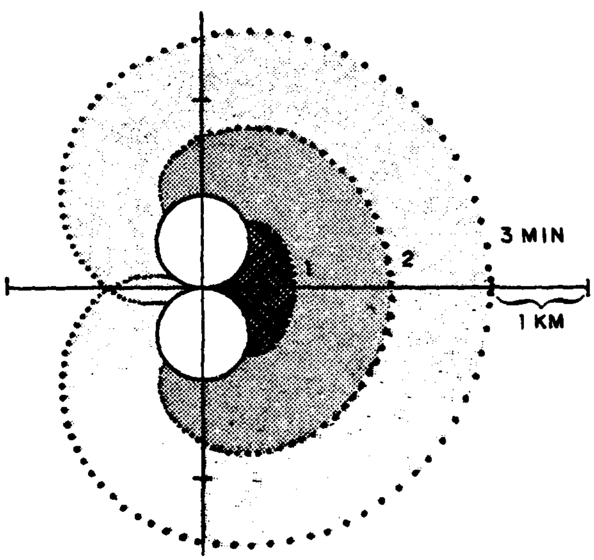


Figure 3. Growth of Target Escape Region with Time (Target Alerted to Attack)

As an example, the darkest shaded area in figure 3 depicts the target escape region 1 minute after alertment. In this figure, the origin of coordinates is defined by the vector $\underline{X}_{to}$ (the assumed target position before alertment).

Because the actual escape region is difficult to describe analytically, it will be approximated by a circumscribing circle as shown in figure 4. Although this approximation introduces errors into subsequent probability computations, these errors decrease with time inasmuch as the actual and approximate escape regions eventually coincide (reference 2). For convenience of notation, R_e is used to denote the radius of the circumscribing circle, and $\hat{\underline{X}}_{ta}$ defines the location of its center relative to $\underline{X}_{to}$.

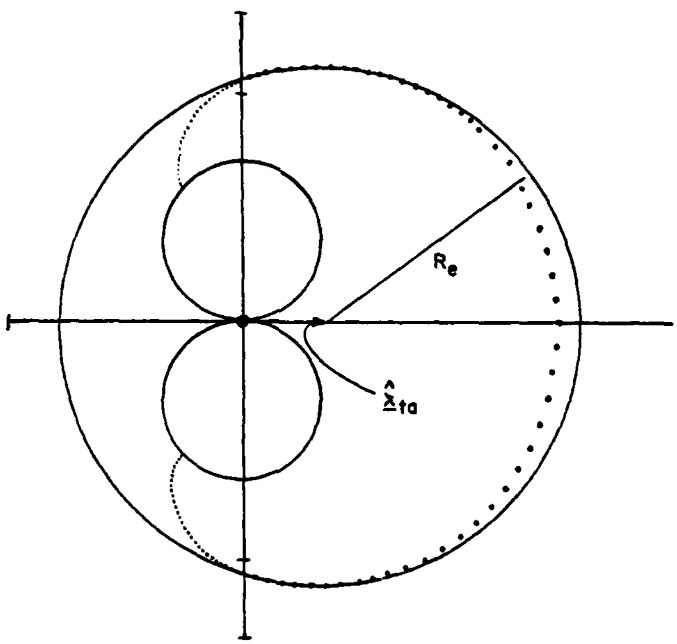


Figure 4. Circular Approximation of Target Escape Region

While it is known that an alerted target must necessarily reside within a prescribed escape region, more specific data to further refine the localization process is usually unavailable. Evasive maneuvers are almost certain to occur because of the crisis nature of the situation (i.e., the target knows it is under attack); however, the absence of additional information precludes classifying one type of maneuver as more likely than another. In light of this, it is not unreasonable to assume that the errors associated with estimating $\underline{X}_{ta}$ are uniformly distributed over the prescribed escape region. Under such an assumption, it can be shown that

$$E\{\underline{X}_{ta}\} = \hat{\underline{X}}_{ta} \tag{4a}$$

$$E\{(\underline{X}_{ta} - \hat{\underline{X}}_{ta})(\underline{X}_{ta} - \hat{X}_{ta})'\} = (R_e/2)^2 I \tag{4b}$$

where I is the two-dimensional identity matrix, and the pertinent statistical computations are performed using a circular approximation as depicted in figure 4.

To determine the probability of target damage resulting from a missile attack, it is necessary to mathematically combine the statistical information embodied in equation sets (1) through (4) with known weapon damage characteristics. This task may be accomplished via application of the law of total probability (reference 1), which leads to the expression

$$\text{Prob}[\text{target damage}] = \int_{\Omega} \text{Prob}[\text{target damage} \mid R]\, p(R)\, dR \tag{5}$$

where

$R = |\underline{X}|$ = range from target to weapon at time of detonation

and

Ω = set of all possible values of R.

Here, the conditional probability function, Prob[target damage|R], defines the probability of target damage given that the relative range from weapon to target is known at time of detonation. It is this term that incorporates weapon damage characteristics into the effectiveness prediction model. The remaining term, p(R), represents the probability density of R and is used to account for target localization errors, target evasion characterisitcs, and weapon aimpoint delivery errors. To see this, observe from figure 2 that the vector $\underline{X}$ can be written as

$$\underline{X} = \underline{X}_{to} + \underline{X}_{ta} - \underline{X}_{w} \ . \tag{6}$$

Thus, both the magnitude and direction of $\underline{X}$ will be affected by target localization errors via $\underline{X}_{to}$, target evasion characteristics via $\underline{X}_{ta}$, and weapon placement errors via $\underline{X}_{w}$.

To compute the probability of target damage on-line and in real time, it is first necessary to explicitly represent p(R) and Prob[target damage|R] as integrable functions of R. Unfortunately, it is virtually impossible to develop exact mathematical representations for these two functions which are also amenable to closed-form integration. Nevertheless, suitable approximations can be employed, thus allowing equation (5) to be evaluated without recourse to numerical integration. The desired approximations for p(R) and Prob[target damage|R] are computed in the following section.

III. APPROXIMATION OF THE PROBABILITY DISTRIBUTIONS

DETERMINATION OF THE PROBABILITY DENSITY, p(R)

To mathematically describe the probability density, p(R), it is convenient to utilize the concept of characteristic functions (reference 1). In particular, if $p(\underline{x})$ defines the probability density of a two-dimensional random vector $\underline{x}' = (x, y)$, then this density and its associated characteristic function $\Phi_x(\underline{\omega})$ satisfy the relations

$$p(\underline{x}) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_x(\underline{\omega}) e^{-j\underline{\omega}'\underline{x}} d\underline{\omega} \tag{7a}$$

$$\Phi_x(\underline{\omega}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(\underline{x}) e^{j\underline{\omega}'\underline{x}} d\underline{x} \tag{7b}$$

where $\underline{\omega}$ is a two-dimensional vector of the form $\underline{\omega}' = (\omega_x, \omega_y)$.

For the problem under consideration, observe from equation (6) that the vector $\underline{X}$ is a linear function of $\underline{X}_{to}$, $\underline{X}_{ta}$, and $\underline{X}_w$. Since the errors associated with estimating these latter three vectors are uncorrelated, it follows (from reference 1) that the characteristic function of $p(\underline{X})$ statisfies the relation

$$\Phi_X(\underline{\omega}) = \Phi_{to}(\underline{\omega})\Phi_{ta}(\underline{\omega})\Phi_w(-\underline{\omega}) \tag{8}$$

where $\Phi_{to}(\underline{\omega})$, $\Phi_{ta}(\underline{\omega})$, and $\Phi_w(\underline{\omega})$ are characteristic functions affiliated with the probability densities $p(\underline{X}_{to})$, $p(\underline{X}_{ta})$, and $p(\underline{X}_w)$, respectively. In view of the assumption that $\underline{X}_{to}$ and $\underline{X}_w$ are

Gaussian-distributed, the functions $\Phi_{to}(\underline{\omega})$ and $\Phi_w(\underline{\omega})$ take the form (from reference 1)

$$\Phi_{to}(\underline{\omega}) = \exp(j\underline{\omega}'\underline{\hat{X}}_{to} - \frac{1}{2}\underline{\omega}'P_{to}\underline{\omega}) \tag{9}$$

$$\Phi_w(\underline{\omega}) = \exp(j\underline{\omega}'\underline{\hat{X}}_w - \frac{1}{2}\underline{\omega}'P_w\underline{\omega}) \ . \tag{10}$$

As discussed previously, the vector $\underline{X}_{ta}$ is presumed to be uniformly distributed over the target escape region; thus,

$$p(\underline{X}_{ta}) = \begin{cases} \dfrac{1}{\pi R_e^2} & |\underline{X}_{ta} - \underline{\hat{X}}_{ta}| \le R_e \\ 0 & |\underline{X}_{ta} - \underline{\hat{X}}_{ta}| > R_e \end{cases} \ . \tag{11}$$

Although this expression may be utilized without alteration, it will facilitate subsequent computations to approximate $p(\underline{X}_{ta})$ via the formula

$$p(\underline{X}_{ta}) \approx p_n(\underline{X}_{ta}) = \left[\frac{n+1}{n}\right]\left[\frac{\exp\{-(n+1)\rho^2\}}{\pi R_e^2}\right]\sum_{k=0}^{n-1}\frac{(n+1)^k}{k!}\rho^{2k} \ . \tag{12a}$$

Here, n determines the order of the approximating series, and

$$\rho = \frac{|\underline{X}_{ta} - \underline{\hat{X}}_{ta}|}{R_e} \ . \tag{12b}$$

The development of the finite-sum Gaussian distributions described in equation set (12) is derived in appendix A. They are often used

to evaluate complicated probability integrals in terms of elementary functions. It might be noted that application of equation set (12), in lieu of equation (11), does not change the mean and covariance of $\underline{X}_{ta}$. Consequently, equation set (4) will continue to provide a valid statistical description of $\underline{X}_{ta}$ regardless of which density function is chosen to approximate the pertinent error distribution. For the case n=1, observe that $p_1(\underline{X}_{ta})$ is Gaussian and reduces to

$$p_1(\underline{X}_{ta}) = \frac{2}{\pi R_e^2} \exp(- \frac{2}{R_e^2} |\underline{X}_{ta} - \hat{\underline{X}}_{ta}|^2) \ . \tag{13}$$

Finally, the limiting approximation as n approaches infinity is the original uniform distribution; i.e.,

$$\lim_{n\to\infty} p_n(\underline{X}_{ta}) = p(\underline{X}_{ta}) \ . \tag{14}$$

The characteristic function for $p_n(\underline{X}_{ta})$, developed in appendix B, is given by

$$\Phi_{ta}(\underline{\omega};n) = \frac{1}{n} L_{n-1}^{(1)} \left[\frac{1}{2}\rho_e \omega^2\right] \exp(j\underline{\omega}'\hat{\underline{X}}_{ta} - \frac{1}{2} \rho_e \omega^2) \tag{15a}$$

where

$$\omega = ||\underline{\omega}|| = \sqrt{\omega_x^2 + \omega_y^2} \tag{15b}$$

$$\rho_e = \frac{R_e^2}{2(n+1)} \tag{15c}$$

and $L_m^{(p)}[\xi]$ are Laguerre polynomials (from reference 2), defined by

$$L_m^{(p)}\,[\xi] = \sum_{k=0}^{m} \binom{m+p}{m-k} \frac{1}{k!}\,(-\xi)^k \qquad m,p=0,1,2,\ldots \tag{16}$$

Here, n appears as an argument in the characteristic function of $p_n(\underline{X}_{ta})$ to denote the order of the approximating series.

Since equations (8), (9), (10), and (15a) completely specify the characteristic function of $p(\underline{X})$, this probability density may be determined analytically from equation (7a) as follows:

$$p(\underline{X}) = \frac{1}{(2\pi)^2} \int_{\infty}^{\infty}\int_{\infty}^{\infty} \Phi_X(\underline{\omega})e^{-j\underline{\omega}'\underline{X}}d\underline{\omega}\ . \tag{17}$$

To obtain the distribution of range errors, recall that $R=|\underline{X}|$. Consequently, if the cartesian coordinates of $\underline{X}$ are given by $\underline{X}'$ = (X,Y), the associated polar coordinates may be expressed as

$$R = \sqrt{X^2 + Y^2} \tag{18a}$$

$$\Theta = \tan^{-1}(Y/X) \quad \text{where} \quad 0 \le \Theta \le 2\pi\ . \tag{18b}$$

The Jacobian of the transformation from cartesian to polar coordinates then takes the form

$$J = \det \begin{bmatrix} \frac{\partial X}{\partial R} & \frac{\partial Y}{\partial R} \\ \frac{\partial X}{\partial \Theta} & \frac{\partial Y}{\partial \Theta} \end{bmatrix} = \det \begin{bmatrix} \cos\Theta & -R\sin\Theta \\ \sin\Theta & R\cos\Theta \end{bmatrix} = R. \tag{19}$$

Using the transformation theorem for probability density functions, it can be shown (from reference 3) that

$$p(R,\Theta) = p(\underline{X})||J||$$

$$= \frac{R}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_X(\underline{\omega}) e^{-j\omega R\cos(\Theta-\Phi)} \, d\underline{\omega} \tag{20}$$

where

$$\Phi = \tan^{-1} (\omega_y/\omega_x) \ . \tag{21}$$

The desired probability density, p(R), follows by integrating $p(R,\Theta)$ over the interval $0 \leq \Theta \leq 2\pi$; i.e.,

$$p(R) = \int_0^{2\pi} p(R,\Theta)d\Theta$$

$$= \frac{R}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_X(\underline{\omega}) \left[\int_0^{2\pi} e^{-j\omega R\cos(\Theta-\Phi)} \, d\Theta \right] d\underline{\omega} \tag{22}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_X(\underline{\omega}) R \ J_0(\omega R) d\underline{\omega}$$

where $J_0(\omega R)$ is a zeroth order Bessel function (from reference 2).

DETERMINATION OF CONDITIONAL TARGET DAMAGE PROBABILITY,
Prob[target damage|R]

To mathematically describe the probability function Prob[target damage|R] requires explicit knowledge of the weapon damage characteristics. For this study, missiles with circular interdiction zones are used as an example to determine the probability of target damage. These characteristics will be specified by concentric circular regions as shown in figure 5 and defined as follows:

1. Lethality or Kill (Seaworthiness Impairment). Lethality from underwater explosions is associated with overpressure necessary to rupture the submarine pressure hull.

2. Immobilization (Mobility Impairment). Immobilization is associated with shock damage to internal equipment. It is defined as that condition whereby the target is rendered virtually immobile because of damage to propulsion machinery.

3. Mission Abort (Weapon Delivery Impairment). Weapon delivery impairment is associated with shock damage to internal equipment involved in target detection and weapon delivery. It is defined as that condition whereby the target is virtually unable to detect enemy vessels or release weapons effectively.

Each region is associated with a 50 percent probability of target damage, and the respective centers all coincide with weapon position at time of detonation. Note also that regions of lesser potential destruction always encompass regions of greater potential destruction as subsets. For computational convenience, R_d will be used to parametrically define the radius of the pertinent damage envelope

under consideration. In this way, the probability of the missile inflicting any prescribed type of damage to a target can be determined from a single formula by simply assigning the appropriate value to R_d.

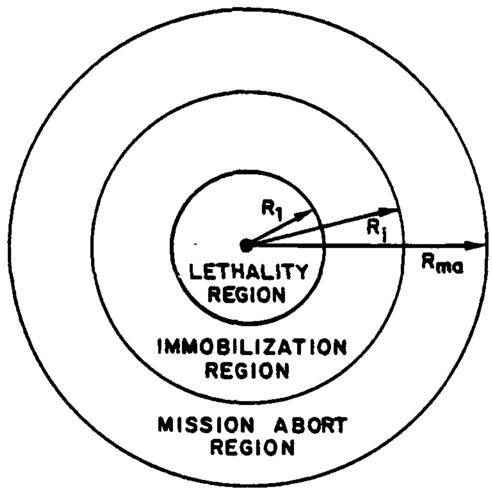


Figure 5. Geometric Description of Missile Damage Characteristics

Although specific numerical data are available, an exact mathematical description of Prob[target damage|R] does not exist. However, from practical considerations it is apparent that this function will decrease monotonically as R increases, and also must satisfy the inequality

$$0 \leq \text{Prob}[\text{target damage}|R] \leq 1 \tag{23}$$

for $0 \leq R < \infty$. These characteristics suggest that finite-sum Gaussian distributions again be used as a basis for approximating Prob[target damage|R]; i.e.,

$$\text{Prob}[\text{target damage}|R] \approx \text{Prob}[\text{target damage}|R,m] \tag{24}$$

where

$$\text{Prob}[\text{target damage}|R,m] = \exp(-m\xi^2) \sum_{k=0}^{m-1} \frac{m^k}{k!} \xi^{2k}. \tag{25}$$

As before, m determines the order of the approximating series, and

$$\xi = R/R_d \; . \tag{26}$$

Graphs of equation (25) for $m=1,2,3,\ldots10$ and $m=\infty$ are plotted in figure 6. As expected, these functions all decrease monotonically as R increases. Note also that the limiting conditional probability may be expressed in the form

$$\text{Prob}[\text{target damage}|R,\infty] = \begin{cases} 1 & 0 \leq R \leq R_d \\ 0 & R > R_d \end{cases} \; . \tag{27}$$

Because of its simplicity, equation (27) is frequently used in practical applications even though it is somewhat artificial. Indeed, such an approximation implicitly assumes that missile attacks will have uniformly devastating effects over the entire weapon damage envelope. In light of this, it is perhaps more realistic, and certainly more economical, to employ lower-order approximations when possible. Accordingly, the parameter m will not be specified here so as to enhance modelling flexibility and realism.

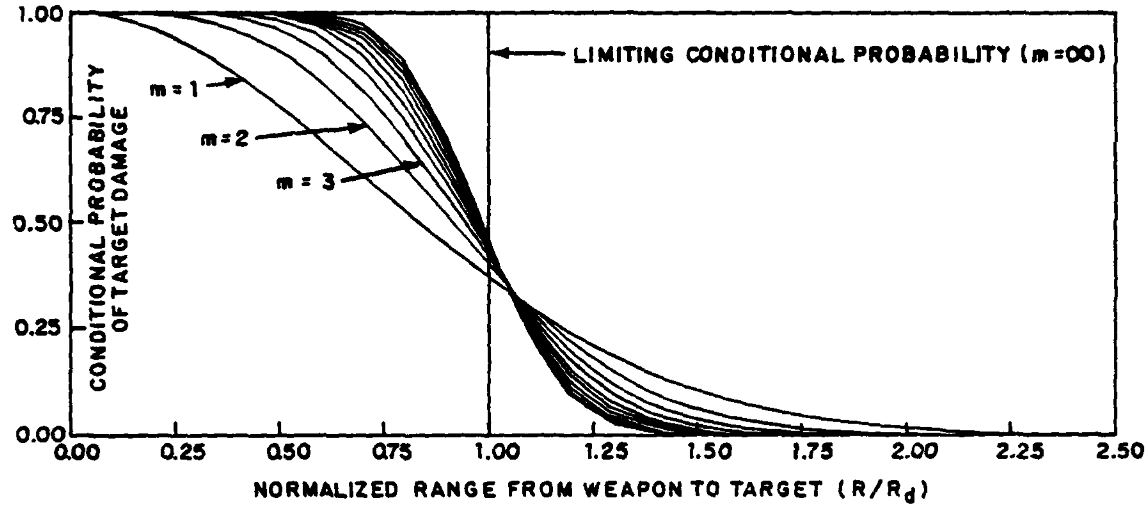


Figure 6. Plots of the Function - Prob[target damage|R,m]

IV. COMPUTATION OF THE TARGET DAMAGE PROBABILITY

The probability of target damage resulting from a missile attack may now be determined analytically by utilizing approximations developed in section III in conjunction with the law of total probability. More precisely, substituting equations (22), (24), (25), and (26) into equation (5), and evaluating the resultant integral over Ω, (see appendix C) leads to the expression

$$\text{Prob}\left[\text{target damage}\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_{X}(\underline{\omega})\Psi(\underline{\omega})d\underline{\omega} \qquad (28)$$

where

$$\Psi(\underline{\omega}) = \frac{\rho_d}{2\pi} L_{m-1}^{(1)}\left[\frac{1}{2}\rho_d\omega^2\right]\exp(-\frac{1}{2}\rho_d\omega^2) \qquad (29a)$$

$$\rho_d = R_d^2/2m \qquad (29b)$$

and $L_{m-1}^{(1)}[\xi]$ is the Laguerre polynomial defined previously. In deriving equation (28), it might be noted that integration of the exact probability distributions over Ω is mathematically equivalent to integrating the approximate distributions over a semi-infinite interval, $0 < R < \infty$.

If equations (8), (9), (10), (15a), and (29a) are combined, the integrand in equation (28) may be rewritten as

$$\Phi_X(\underline{\omega})\Psi(\underline{\omega}) = \frac{\rho_d}{2\pi n} L_{n-1}^{(1)}\left[\frac{1}{2}\rho_e\omega^2\right] L_{m-1}^{(1)}\left[\frac{1}{2}\rho_d\omega^2\right]\exp\left(j\underline{\omega}'\hat{\underline{X}} - \frac{1}{2}\underline{\omega}'P_{\underline{x}}\underline{\omega}\right) \qquad (30)$$

where

$$\underline{\hat{X}} = \begin{bmatrix} \hat{X} \\ \hat{Y} \end{bmatrix} = \underline{\hat{X}}_{to} + \underline{\hat{X}}_{ta} - \underline{\hat{X}}_{w} \tag{31a}$$

$$P_x = \begin{bmatrix} P_x(1,1) & P_x(1,2) \\ P_x(1,2) & P_x(2,2) \end{bmatrix} = P_{to} + P_w + (\rho_e + \rho_d)I \tag{31b}$$

and I is a (2x2) identity matrix. Utilizing equation (16), in conjunction with equations (28) and (30), then yields

$$\mathrm{Prob}\left[\text{target damage}\right] = \frac{\rho_d}{n} \sum_{k=0}^{n-1} \sum_{i=0}^{m-1} \binom{n}{k+1}\binom{m}{i+1} \rho_e^k \, \rho_d^i \Omega_{k+1} \tag{32}$$

where

$$\Omega_{k+i} = \frac{1}{2\pi} \frac{(-1)^{k+i}}{k!\ i!} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{\omega^2}{2}\right)^{k+i} \exp(j\underline{\omega}'\underline{\hat{X}} - \tfrac{1}{2}\underline{\omega}'P_x\underline{\omega})d\underline{\omega} \ . \tag{33}$$

The last remaining task is to determine Ω_{k+i} in terms of the pertinent modeling parameters. This may be accomplished by employing equation (15b) and the Binomial Expansion Theorem to explicitly evaluate the integral appearing in equation (33). Pertinent mathematical details are presented in appendix D, and the final result takes the form

$$\Omega_{k+i} = \frac{\exp\{-\frac{1}{2}(\hat{\xi}_x^2 + \hat{\xi}_y^2)\}}{\sqrt{\lambda_+ \ \lambda_-}} \binom{k+i}{k} \sum_{\ell=0}^{k+i} \Theta_\ell\left[\hat{\xi}_x, \lambda_+\right]\Theta_{k+i-\ell}\left[\hat{\xi}_y, \lambda_-\right] \tag{34}$$

$$\Theta_{\ell}[\xi,\lambda] = \binom{2\ell}{\ell} [-4\lambda]^{-\ell} \sum_{q=0}^{\ell} \binom{\ell}{q} \frac{[-\xi^2]^q}{(2q-1)!!} \tag{35}$$

where $\lambda_{\pm}$ are the eigenvalues of P_x, given by

$$\lambda_{\pm} = \left[\frac{P_x(1,1)+P_x(2,2)}{2}\right] \pm \sqrt{\left[\frac{P_x(1,1)-P_x(2,2)}{2}\right]^2 + P_x^2(1,2)} \tag{36}$$

and

$$\begin{bmatrix} \hat{\xi}_x \\ \hat{\xi}_y \end{bmatrix} = \begin{bmatrix} \dfrac{\sin\phi}{\sqrt{\lambda_+}} & \dfrac{\cos\phi}{\sqrt{\lambda_+}} \\ -\dfrac{\cos\Theta}{\sqrt{\lambda_-}} & \dfrac{\sin\Theta}{\sqrt{\lambda_-}} \end{bmatrix} \begin{bmatrix} \hat{X} \\ \hat{Y} \end{bmatrix} \tag{37}$$

with

$$\phi = \tan^{-1}\left[\frac{P_x(1,2)}{\lambda_+ - P_x(1,1)}\right]. \tag{38}$$

An algorithm to evaluate equation (32) via digital computer has been written in FORTRAN. A flowchart of this algorithm and a FORTRAN listing are presented in appendix E. The required input data to this algorithm are defined as follows:

1. Target Localization Data

$\begin{bmatrix} XTO(1) \\ XTO(2) \end{bmatrix} = \hat{\underline{X}}_{to}$ estimated target position prior to alertment of a missile attack; this vector is defined with own ship as the origin of coordinates

$\begin{bmatrix} PTO(1,1) & PTO(1,2) \\ PTO(2,1) & PTO(2,2) \end{bmatrix} = P_{to}$ covariance matrix of $\hat{\underline{X}}_{to}$

$\begin{bmatrix} XTA(1) \\ XTA(2) \end{bmatrix} = \hat{\underline{X}}_{ta}$ estimated target position after alertment to a missile attack; this vector is defined with $\hat{X}_{to}$ as the origin of coordinates

$RE = R_e$ evasion radius of target

$N = n$ integer which determines the order of the Gaussian sum approximation to $p(\underline{X}_{ta})$

2. Weapon Placement Data

$\begin{bmatrix} XW(1) \\ XW(2) \end{bmatrix} = \hat{\underline{X}}_w$ estimated weapon position at time of detonation; this vector is defined with own ship as the origin of coordinates

$\begin{bmatrix} PW(1,1) & PW(1,2) \\ PW(2,1) & PW(2,2) \end{bmatrix} = P_w$ covariance matrix of $\underline{X}_w$

3. Weapon Damage Characteristics

RD = R_d damage radius of weapon

M = m integer which determines the order of the Gaussian sum approximation to Prob[target damage|R]

Once the preceding data have been specified, the algorithm will compute PROB = probability of target damage each time the subroutine missile (PROB) is called.

V. SUMMARY

A real-time procedure for predicting the effectiveness of a missile attack against prescribed target threats has been developed. The required probability of target damage is specified in terms of known or measurable weapon/target characteristics and associated error statistics. Furthermore, the computational algorithm is sufficiently compact so as to allow weapon performance predictions to be updated on-line as tactical conditions change. As a result, the algorithm has potential value as an automatic decision aid to assist in the optimal deployment of missiles.

Although the weapon damage characteristics and target localization solution are constant inputs to the probability model, other inputs (e.g., missile aimpoint coordinates and target evasion characteristics) are time-dependent functions which change as the missile aimpoint and flight time are varied. In view of these properties, it is evident that an optimum aimpoint exists which will maximize targt damage probability for any prescribed tactical scenario. Future work will focus on application of stochastic optimization techniques to solve the non-linear equations which determine the optimum missile aimpoint.

APPENDIX A

DEVELOPMENT OF FINITE-SUM GAUSSIAN DISTRIBUTIONS

Given a uniform density of the form

$$p(\underline{X}_{ta}) = \begin{cases} \dfrac{1}{\pi R^2} & |\underline{X}_{ta} - \hat{\underline{X}}_{ta}| \leq R_e \\ 0 & |\underline{X}_{ta} - \hat{\underline{X}}_{ta}| > R_e \end{cases} \tag{A-1a}$$

where $$\underline{X}_{ta} = \begin{bmatrix} x \\ y \end{bmatrix} \tag{A-1b}$$

and $$\hat{\underline{X}}_{ta} = E\left[\underline{X}_{ta}\right] = \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}, \tag{A-1c}$$

it is desired to approximate it by a finite-sum of Gaussian distributions of the form

$$p(\underline{X}_{ta}) \approx \hat{p}_n(\underline{X}_{ta}) = K \exp\left\{-\alpha|\underline{X}_{ta} - \hat{\underline{X}}_{ta}|^2\right\} \sum_{i=o}^{n-1} \frac{\alpha^i |\underline{X}_{ta} - \hat{\underline{X}}_{ta}|^{2i}}{i!} . \tag{A-2}$$

In the approximating series, n determines the order of the series. As n approaches infinity

$$\hat{p}_\infty(\underline{X}_{ta}) = p(\underline{X}_{ta}). \tag{A-3}$$

The constants, K and α in equation (A-2) are found by evaluating the

statistics of the approximating density so that the mean and covariance between the two densities remain unchanged. Also, for $\hat{p}_n(\underline{X}_{ta})$ to be a valid probability density it must satisfy the following constraints:

$$\hat{p}_n(\underline{X}_{ta}) \geq 0 \tag{A-4a}$$

and

$$\int_{-\infty}^{\infty} \hat{p}_n(\underline{X}_{ta})\, d\underline{X}_{ta} = 1. \tag{A-4b}$$

Before constraint (A-4a) can be satisfied, the constants K and α must be found. First, it will be shown that the approximating probability density satisfies constraint (A-4b). In order to simplify the integration involved in satisfying the constraint, a transformation from rectangular to polar coc:dinates is necessary. Using figure A-1 let

$$x-\hat{x} = R \cos \Theta$$

and

$$y-\hat{y} = R \sin \Theta$$

or

$$R^2 = (x-\hat{x})^2 + (y-\hat{y})^2$$

and

$$\Theta = \tan^{-1} \left\{ (y-\hat{y})/(x-\hat{x}) \right\} .$$

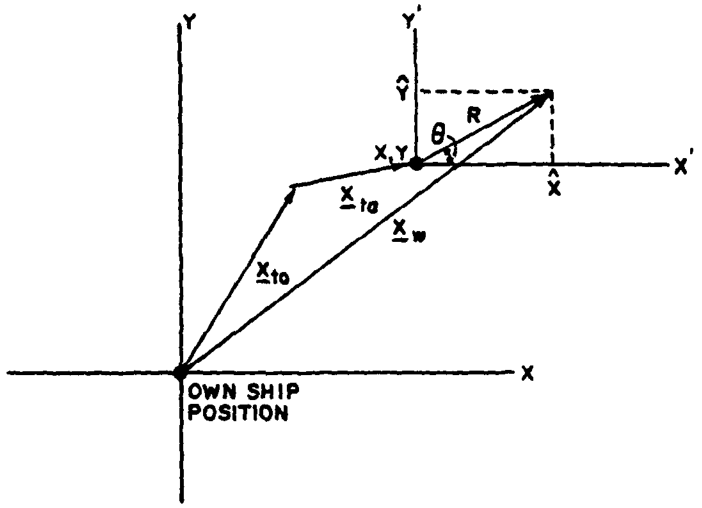


Figure A-1. Relationship Between Polar and Rectangular Coordinates of the Actual and Estimated Target Position

Substituting for x and y into equation (A-2) yields

$$\tilde{p}_n(\underline{X}_{ta}) = \tilde{p}_n(x,y) = \tilde{p}_n(R,\Theta) = Ke^{-\alpha R^2} \sum_{i=o}^{n-1} \frac{\alpha^i R^{2i}}{i!} . \tag{A-5}$$

From reference A-1, the integration in polar coordinates becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{p}_n(\underline{X}_{ta})d\underline{X}_{ta} = K \sum_{i=o}^{n-1} \frac{\alpha^i}{i!} \int_o^{\infty}\int_o^{2\pi} e^{-\alpha R^2} R^{2i} \left| \frac{\partial(x,y)}{\partial(R,\Theta)} \right| dR d\Theta = 1$$

where the Jacobian of the transformation is defined by

$$J = \left|\frac{\partial(x,y)}{\partial(R,\Theta)}\right| = \det \begin{bmatrix} \frac{\partial x}{\partial R} & \frac{\partial y}{\partial R} \\ \frac{\partial x}{\partial \Theta} & \frac{\partial y}{\partial \Theta} \end{bmatrix} = R.$$

Integrating over Θ, the equation reduces to

$$2\pi K \sum_{i=o}^{n-1} \frac{\alpha^i}{i!} \int_o^{\infty} e^{-\alpha R^2} R^{2i+1} \, dR = 1. \qquad \text{(A-6)}$$

From reference A-2 (p.65, #314, 2b) the integral in equation (A-6) becomes

$$2\pi K \sum_{i=o}^{n-1} \frac{\alpha^i}{i!} \left[\frac{i!}{2\alpha^{i+1}}\right] = \frac{\pi K}{\alpha} \sum_{i=o}^{n-1} 1 = 1 \; . \qquad \text{(A-7)}$$

The summation in equation (A-7) can be written as

$$\sum_{i=o}^{n-1} 1 = 1+1+1+1.....1 = n. \qquad \text{(A-8)}$$

Substituting equation (A-8) into equation (A-7) and solving for K yields

$$K = \alpha/\pi n. \qquad \text{(A-9)}$$

Selecting the constant K by equation (A-9) insures that the constraint given by (A-4b) is satisfied.

Next the mean of the approximating density is found. The mean is given by

$$\text{mean} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underline{X}_{ta}\, \tilde{p}_n(\underline{X}_{ta})\, d\underline{X}_{ta}$$

which can be written as

$$\text{mean} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\underline{X}_{ta} - \hat{\underline{X}}_{ta})\tilde{p}_n(\underline{X}_{ta})\, d\underline{X}_{ta} + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{\underline{X}}_{ta}\, \tilde{p}_n(\underline{X}_{ta}) d\underline{X}_{ta}. \quad \text{(A-10)}$$

Utilizing constraint (A-4a), the second term reduces to

$$\hat{\underline{X}}_{ta} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{p}_n(X_{ta}) d\underline{X}_{ta} = \hat{\underline{X}}_{ta}. \quad \text{(A-11)}$$

To evaluate the first integral, the substitutions

$$x'_{ta} = x_{ta} - \hat{x}_{ta}$$

$$dx'_{ta} = dx_{ta}$$

$$y'_{ta} = y_{ta} - \hat{y}_{ta}$$

and

$$dy'_{ta} = dy_{ta}$$

are made. The integral now becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\underline{X}_{ta} - \hat{\underline{X}}_{ta})\tilde{p}_n(\underline{X}_{ta})d\underline{X}_{ta} =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \begin{bmatrix} x'_{ta} \\ y'_{ta} \end{bmatrix} \tilde{p}_n(x'_{ta} + \hat{x}_{ta}, y'_{ta} + \hat{y}_{ta})dx'_{ta}\ dy'_{ta} . \qquad \text{(A-12)}$$

Making a transformation to polar coordinates where

$$x'_{ta} = R \cos \Theta$$

$$y'_{ta} = R \sin \Theta$$

results in two integrals of the form

$$\int_{0}^{2\pi}\int_{0}^{\infty} R \cos \Theta\ \tilde{p}_n(x'_{ta} + \hat{x}_{ta}, y'_{ta} + \hat{y}_{ta})\ RdRd\Theta \qquad \text{(A-13)}$$

and

$$\int_{0}^{2\pi}\int_{0}^{\infty} R \sin \Theta\ \tilde{p}_n(x'_{ta} + \hat{x}_{ta}, y'_{ta} + \hat{y}_{ta})\ RdRd\Theta. \qquad \text{(A-14)}$$

By carrying out the actual integration, it is found that both integrals are zero. Therefore, the mean of the approximating probability density function is the same as the uniform probability density as seen by equation (A-11).

The covariance of the uniform density is given by

$$P_{ta} = (R_e^2/2)I \tag{A-15}$$

where I is a 2x2 identify matrix. The constants K and α in the approximating density function will be determined so that its covariance is also P_{ta}. The covariance is given by

$$\text{covariance} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\underline{X}_{ta} - \hat{\underline{X}}_{ta})(\underline{X}_{ta} - \hat{\underline{X}}_{ta})' \tilde{p}_n(\underline{X}_{ta})d\underline{X}_{ta}. \tag{A-16}$$

Making the transformation to polar coordinates, the result is

$$\text{covariance} = \int_0^{2\pi}\int_0^{\infty} \begin{bmatrix} R\cos\Theta \\ R\sin\Theta \end{bmatrix} \begin{bmatrix} R\cos\Theta & R\sin\Theta \end{bmatrix} \tilde{p}_n(R\cos\Theta,\ R\sin\Theta)RdRd\Theta \tag{A-17}$$

which can be written as four integrals of the form

$$\int_0^{2\pi}\int_0^{\infty} R^2\cos^2\Theta\ \tilde{p}_n(R\cos\Theta,\ R\sin\Theta)RdRd\Theta \tag{A-18a}$$

$$\int_0^{2\pi}\int_0^{\infty} R^2\cos\Theta\ \sin\Theta\ \tilde{p}_n(R\cos\Theta,\ R\sin\Theta)R\,dR\,d\Theta \qquad \text{(A-18b)}$$

$$\int_0^{2\pi}\int_0^{\infty} R^2\sin\Theta\ \cos\Theta\ \tilde{p}_n(R\cos\Theta,\ R\sin\Theta)R\,dR\,d\Theta \qquad \text{(A-18c)}$$

$$\int_0^{2\pi}\int_0^{\infty} R^2\sin^2\Theta\ \tilde{p}_n(R\cos\Theta,\ R\sin\Theta)R\,dR\,d\Theta\ . \qquad \text{(A-18d)}$$

Using the trigonometric identity,

$$\sin\Theta\ \cos\Theta = \frac{1}{2}\sin 2\Theta,$$

the integrals given by equations (A-18b) and (A-18c) are zero. The integrals given by equation (A-18a) and (A-18d) can be easily evaluated by substituting equations (A-5) and (A-9) for $\tilde{p}(R\cos\Theta,\ R\sin\Theta)$. After the substitution, equations (A-18a) and (A-18d) become

$$\frac{\alpha}{\pi n}\ \sum_{i=o}^{n-1}\ \frac{\alpha^i}{i!}\ \int_0^{\infty}\int_0^{2\pi} e^{-\alpha R^2} R^{2i+3}\ \cos^2\Theta\ d\Theta\, dR \qquad \text{(A-19a)}$$

and

$$\frac{\alpha}{\pi n} \sum_{i=o}^{n-1} \frac{\alpha^i}{i!} \int_o^\infty \int_o^{2\pi} e^{-\alpha R^2} R^{2i+3} \sin^2\theta \, d\theta dR. \tag{A-19b}$$

Integrating with respect to θ, both equations (A-19a) and (A-19b) reduce to

$$\frac{\alpha}{n} \sum_{i=o}^{n-1} \frac{\alpha^i}{i!} \int_o^\infty e^{-\alpha R^2} R^{2(i+1)+1} dR. \tag{A-19c}$$

From reference A-2 (p. 65, #314, 2b), the integral becomes

$$\frac{\alpha}{n} \sum_{i=o}^{n-1} \frac{\alpha^i}{i!} \frac{(i+1)!}{2\alpha^{i+2}} = \frac{1}{2\alpha n} \sum_{i=o}^{n-1} i+1. \tag{A-20a}$$

The summation can be written as (from reference A-2) as

$$\sum_{i=o}^{n-1} i+1 = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}. \tag{A-20b}$$

Substituting equation (A-20b) into equation (A-20a) yields

$$\frac{1}{2\alpha n} \sum_{i=o}^{n-1} i+1 = \frac{n+1}{4\alpha}. \tag{A-20c}$$

Since the covariance of the uniform density and the approximating density must be the same, equation (A-20c) is set equal to equation (A-15) (covariance matrix of the uniform density), and the constant α is determined. The result is

$$\alpha = (n+1)/R_e^2 . \tag{A-21}$$

Substituting equation (A-21) into equation (A-9), the constant K is given by

$$K = \left[\frac{n+1}{n}\right] \frac{1}{\pi R_e^2} \tag{A-22}$$

and the approximating density, equation (A-2), becomes

$$\tilde{p}_n(\underline{X}_{ta}) = \frac{n+1}{n} \; \frac{\exp\left\{\dfrac{-(n+1)\ |\underline{X}_{ta} - \hat{\underline{X}}_{ta}|^2}{R_e^2}\right\}}{\pi R_e^2} \sum_{i=o}^{n-1} \frac{(n+1)^i |\underline{X}_{ta} - \hat{\underline{X}}_{ta}|^{2i}}{i!\ R_e^{2i}} . \tag{A-23}$$

Since n, R_e, and $|\underline{X}_{ta} - \hat{\underline{X}}_{ta}|$ are all greater than or equal to zero,

$$\tilde{p}(\underline{X}_{ta}) > 0$$

and the constraint given by equation (A-4a) is satisfied.

APPENDIX B

DEVELOPMENT OF THE CHARACTERISTIC FUNCTION

Given a probability density of the form

$$\tilde{P}_n(\underline{X}_{ta}) = \frac{n+1}{n} \; \frac{\exp\left\{\dfrac{-(n+1)\,|\underline{X}_{ta} - \hat{\underline{X}}_{ta}|^2}{R_e^2}\right\}}{\pi R_e^2} \sum_{i=o}^{n-1} \frac{(n+1)^i |\underline{X}_{ta} - \hat{\underline{X}}_{ta}|^{2i}}{i!\; R_e^{2i}} \qquad \text{(B-1)}$$

the characteristic function can be found from the equation

$$\Phi_{ta}(\underline{\omega}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_n(\underline{X}_{ta})\, e^{j\underline{\omega}'\underline{X}_{ta}}\, d\underline{X}_{ta} \qquad \text{(B-2)}$$

where

$$\underline{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \end{bmatrix} \text{and } \underline{\omega}' = \begin{bmatrix} \omega_x, \omega_y \end{bmatrix}.$$

Substituting equation (B-1) into equation (B-2) and letting

$$\underline{X} = \underline{X}_{ta} - \hat{\underline{X}}_{ta}$$

$$d\underline{X} = d\underline{X}_{ta}$$

results in a characteristic function of the form

$$\Phi_{ta}(\underline{\omega}) = e^{j\underline{\omega}'\hat{\underline{X}}_{ta}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_n(\underline{X} + \hat{\underline{X}}_{ta})\, e^{j\underline{\omega}'\underline{X}}\, d\underline{X}. \qquad (B-3)$$

Making a transformation to polar coordinates, where

$$\underline{\omega} = \begin{bmatrix} \omega \cos\phi \\ \omega \sin\phi \end{bmatrix} \qquad \text{and} \qquad \underline{X} = \begin{bmatrix} R \cos\Theta \\ R \sin\Theta \end{bmatrix}$$

$$\omega = ||\underline{\omega}|| \qquad\qquad R = ||\underline{X}||$$

the characteristic equation is given by

$$\Phi_{ta}(\underline{\omega}) = e^{j\underline{\omega}'\hat{\underline{X}}_{ta}} \left[\frac{n+1}{n}\right] \frac{1}{\pi R_e^2} \int_{o}^{\infty} \exp\left\{-(n+1)\left(\frac{R}{R_e}\right)^2\right\} \sum_{i=o}^{n-1} \frac{(n+1)^i}{i!} \left(\frac{R}{R_e}\right)^{2i}$$

$$\cdot \int_{o}^{2\pi} \exp\,\{jR\omega\cos(\Theta-\phi)\}\, RdRd\Theta, \qquad (B-4)$$

where R_e is the radius of the evasion circle. Integrating with respect to Θ leaves

$$\Phi_{ta}(\underline{\omega}) = 2e^{j\underline{\omega}'\acute{\underline{X}}_{ta}} \left[\frac{n+1}{n}\right] \sum_{k=o}^{n-1} \frac{(n+1)^k}{k!}$$

$$\cdot \int_{o}^{\infty} \exp\left\{-(n+1)\ (R/R_e^2)\right\} (R/R_e)^{2k+1}\, J_o\left(\omega R_e(R/R_e)\right) \frac{dR}{R_e}, \qquad (B-5)$$

where $J_o\left\{\omega R_e(R/R_e)\right\}$ is the zeroth order Bessel function (reference 7).

Let $\rho_e = R/R_e$, then equation (B-5) becomes

$$\Phi_{ta}(\underline{\omega}) = e^{j\underline{\omega}'\hat{\underline{X}}_{ta}} \frac{1}{n} \sum_{k=o}^{n-1} \frac{(n+1)^k}{k!}$$

$$\cdot \int_o^{\infty} \rho_e^{2k} \exp\left\{-(n+1)\rho_e^2\right\} J_o\,(\omega R_e \rho_e)\,(n+1)2\rho_e d\rho_e . \qquad \text{(B-6)}$$

Making another substitution, $\lambda = (n+1)\rho_e^2$, equation (B-6) becomes

$$\Phi_{ta}(\underline{\omega}) = \frac{1}{n} e^{j\underline{\omega}'\hat{\underline{X}}_{ta}} \sum_{k=o}^{n-1} \frac{1}{k!} \int_o^{\infty} \lambda^k e^{-\lambda} J_o\left(2\sqrt{\frac{\omega^2 R_e^2 \lambda}{4(n+1)}}\right) d\lambda$$

which integrates to (from reference A-2, p. 721, #6.643,4)

$$\Phi_{ta}(\omega) = \frac{1}{n} \exp\left\{j\underline{\omega}'\hat{\underline{X}}_{ta} - \frac{\omega^2 R_e^2}{4(n+1)}\right\} \sum_{k=o}^{n-1} L_k^{(o)}\left[\frac{\omega^2 R_e^2}{4(n+1)}\right] \qquad \text{(B-7)}$$

where $L_k^{(o)}$ are Laguerre polynomials (reference 7). By noting (from reference A-2, p. 1038, #8.974,3)

$$\sum_{k=o}^{n-1} L_k^{(o)}\left[\frac{\omega^2 R_e^2}{4(n+1)}\right] = L_{n-1}^{(1)}\left[\frac{\omega^2 R_e^2}{4(n+1)}\right] \tag{B-8}$$

equation (B-7) becomes

$$\Phi_a(\underline{\omega}) = \frac{1}{n}\exp\left\{j\underline{\omega}'\underline{X}_{ta} - \frac{\omega^2 R_e^2}{4(n+1)}\right\} L_n^{(1)}\left[\frac{\omega^2 R_e^2}{4(n+1)}\right]. \tag{B-9}$$

APPENDIX C

PROBABILITY OF TARGET DAMAGE

To determine the probability of target damage, we use the expression

$$\text{Prob}[\text{target damage}] = \int_{\Omega} \text{Prob}[\text{target damage} \mid R]\; p(R)\; dr \qquad \text{(C-1)}$$

where

$R = |X|$ = range from target to weapon at time of detonation

Ω = set of all possible values of R.

The conditional probability, Prob[target damage|R], is approximated by a finite-sum Gaussian distribution given by

$$\text{Prob}[\text{target damage} \mid R] \approx \text{Prob}[\text{target damage} \mid R,m]$$

$$= \exp\left\{-\frac{mR^2}{R_d^2}\right\} \sum_{k=o}^{m-1} \frac{m^k}{k!} \left(\frac{R}{R_d}\right)^{2k} \qquad \text{(C-2)}$$

where m determines the order of the approximating series and R_d is the radius of the pertinent damage envelope. The probability density function p(R) is written as

$$p(R) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_x(\underline{\omega}) R \; J_o(\omega R) \; d\underline{\omega} \tag{C-3}$$

where $\Phi_x(\underline{\omega})$ is the characteristic function defined by

$$\Phi_x(\underline{\omega}) = \Phi_{t_o}(\underline{\omega}) \Phi_{ta}(\underline{\omega}) \Phi_w(-\underline{\omega}) \tag{C-4}$$

and $J_o(\omega R)$ is the zeroth order Bessel function (reference 7). Substituting equations (C-2) and (C-3) into equation (C-1) yields

$$\text{Prob}\left[\text{target damage}\right] = \frac{1}{2\pi} \sum_{k=o}^{m-1} \frac{1}{k!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_x(\underline{\omega}) \int_{\Omega} m^k \left(\frac{R}{R_d}\right)^{2k} \exp\left\{-m\left(\frac{R}{R_d}\right)^2\right\}$$

$$\cdot \; J_o(\omega R) R dR d\underline{\omega} \tag{C-5}$$

Let

$$\psi(\underline{\omega}) = \frac{1}{2\pi} \sum_{k=o}^{m-1} \frac{1}{k!} \int_{\Omega} m^k \left(\frac{R}{R_d}\right)^{2k} e^{-m(R/R_d)^2} J_o(\omega R) R dR \Bigg) \tag{C-6a}$$

and carry out the integration. Since the conditional probability distribution is an approximation, it is equivalent to integration over the interval $0 < R < \infty$. Equation (C-6a) then becomes

$$\psi(\underline{\omega}) = \frac{1}{2\pi} \sum_{k=o}^{m-1} \frac{1}{k!} \int_{o}^{\infty} m^k \left(\frac{R}{R_d}\right)^{2k} e^{-m(R/R_d)^2} J_o(\omega R)RdR \; . \qquad \text{(C-6b)}$$

The integral in equation (C-6b) has the same form as equation (B-6). By using the substitution,

$$\lambda = m(R/R_d)^2$$

the solution to equation (C-6b) is given by

$$\psi(\omega) = \frac{1}{2\pi} \sum_{k=o}^{m-1} \frac{R_d^{\;2}}{2m} \exp\left\{\frac{-\omega^2 R_d^{\;2}}{4m}\right\} L_k^{(0)}\left(\frac{\omega^2 R_d^{\;2}}{4m}\right). \qquad \text{(C-7)}$$

Using equation (B-8), equation (C-7) can be written as

$$\psi(\omega) = \frac{1}{2\pi} L_{m-1}^{(1)} \left[\frac{\omega^2 R_d^{\;2}}{4m}\right] \frac{R_d^2}{2m} \exp\left\{\frac{\omega^2 R_d^2}{4m}\right\} \qquad \text{(C-8)}$$

Substituting equation (C-8) into equation (C-5), the probability of target damage now takes the form

$$\text{Prob}\left[\text{target damage}\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_x(\underline{\omega})\; \psi(\underline{\omega}) d\underline{\omega}. \qquad \text{(C-9)}$$

APPENDIX D

EVALUATION OF Ω_{k+i}

Determination of the probability of target damage requires explicit evaluation of the following integral:

$$\Omega_{k+i} = \frac{1}{2\pi} \frac{(-1)^{k+i}}{k!i!} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\frac{\omega^2}{2}\right]^{k+i} \exp\left\{j\underline{\omega}'\hat{\underline{X}} - \frac{1}{2}\underline{\omega}' P_x \underline{\omega}\right\} d\underline{\omega} \qquad \text{(D-1)}$$

where

$$\hat{\underline{X}} = \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \hat{\underline{X}}_{to} + \hat{\underline{X}}_{ta} - \hat{\underline{X}}_{w} ,$$

P_x is a symmetric positive definite covariance matrix of the form

$$P_x = \begin{bmatrix} P_x(1,1) & P_x(1,2) \\ P_x(1,2) & P_x(2,2) \end{bmatrix} = P_{to} + P_w + (\rho_e + \rho_d)I,$$

and I is a (2x2) identity matrix. A diagonal matrix which contains the eigenvalues of P_x can be formed by using the formula

$$\Lambda = M^{-1}P_x M = \text{diag}\,(\lambda_x, \lambda_y) \qquad \text{(D-2)}$$

where M is the modal matrix and λ_x, λ_y are the eigenvalues of P_x. Solving equation (D-2) for P_x yields

$$P_x = M\Lambda M^{-1}. \tag{D-3}$$

Since P_x is symmetric, the eigenvectors are orthogonal and $M^{-1} = M'$. Now equation (D-3) becomes

$$P_x = M\Lambda M'. \tag{D-4}$$

Substituting $N' = M$ and $Q'Q = \Lambda$ into equation (D-4) results in

$$P_x = N'Q'QN. \tag{D-5}$$

Substituting equation (D-5) into equation (D-1), the exponent becomes

$$\exp\left\{j\underline{\omega}'\underline{\hat{X}} - \frac{1}{2}(QN\underline{\omega})'(QN\underline{\omega})\right\}. \tag{D-6}$$

Making another substitution,

$$\underline{\xi} = QN\underline{\omega} \tag{D-7}$$

and using equations (D-5), (D-6) and (D-7), the integral in equation (D-1) reduces to

$$\Omega_{k+i} = \frac{1}{2\pi} \frac{(-1)^{k+i}}{k!i!} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\frac{\underline{\xi}'(Q^{-1})(N^{-1})'N^{-1}Q^{-1}\underline{\xi}}{2}\right]^{k+i}$$

$$\cdot \exp\left\{j\underline{\xi}'(Q^{-1})'(N^{-1})'\hat{\underline{X}} - \frac{1}{2}\underline{\xi}'\underline{\xi}\right\} \left|\frac{\partial(\omega_x,\omega_y)}{\partial(\xi_x,\xi_y)}\right| d\underline{\xi}, \qquad \text{(D-8)}$$

where $\left|\frac{\partial(\omega_x,\omega_y)}{\partial(\xi_x,\xi_y)}\right|$ is the Jacobian of the transformation (reference A-1).

The Jacobian of the transformation is found by writing equation (D-7) as

$$\underline{\omega} = N^{-1}\,Q^{-1}\,\underline{\xi} = \begin{bmatrix} T_{xx} & T_{xy} \\ T_{yx} & T_{yy} \end{bmatrix} \begin{bmatrix} \xi_x \\ \xi_y \end{bmatrix} = T\underline{\xi}$$

where

$$T = N^{-1}Q^{-1}.$$

The partial derivatives take the form

$$\frac{\partial \omega_x}{\partial \xi_x} = T_{xx} \qquad \frac{\partial \omega_x}{\partial \xi_y} = T_{xy}$$

$$\frac{\partial \omega_y}{\partial \xi_x} = T_{yx} \qquad \frac{\partial \omega_y}{\partial \xi_y} = T_{yy},$$

and the Jacobian of the transformation is given by

$$\left| \frac{\partial(\omega_x,\ \omega_y)}{\partial(\xi_x,\ \xi_x)} \right| = \begin{vmatrix} T_{xx} & T_{xy} \\ T_{yx} & T_{yy} \end{vmatrix} = |T| = \frac{1}{|N|\,|Q|} \ . \tag{D-9}$$

The determinants in equation (D-9) are given by

$$|Q| = \sqrt{\lambda_x \lambda_y} \quad \text{and} \quad |N| = 1.$$

The second determinant was defined to be the transpose of the modal matrix, M. Since $M^{-1} = M'$, the determinant of M must be unity; therefore, equation (D-9) becomes

$$\left| \frac{\partial(\omega_x,\omega_y)}{\partial(\xi_x,\xi_y)} \right| = \frac{1}{\sqrt{\lambda_x \lambda_y}} \ . \tag{D-10}$$

Substituting equation (D-10) into equation (D-8) and letting M = N' yields

$$\Omega_{k+i} = \frac{1}{2\pi}\,\frac{(-1)^{k+i}}{k!i!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\underline{\xi}'\Lambda^{-1}\underline{\xi}}{2}\right)^{k+i} \exp\,\{j\underline{\xi}'(Q^{-1})'M'\hat{\underline{X}} - \tfrac{1}{2}\underline{\xi}'\underline{\xi}\}\,\frac{d\underline{\xi}}{\sqrt{\lambda_x\lambda_y}}. \quad \text{(D-11)}$$

Let $\qquad \hat{\underline{\xi}} = (Q^{-1})'M'\hat{\underline{X}} = \begin{bmatrix} \hat{\xi}_x \\ \hat{\xi}_y \end{bmatrix}$

$$\underline{\xi}'\underline{\xi} = \xi_x^2 + \xi_y^2$$

$$\underline{\xi}'\,\Lambda^{-1}\underline{\xi} = \frac{\xi_x^2}{\lambda_x} + \frac{\xi_y^2}{\lambda_y},$$

Equation (D-11) is then written as

$$\Omega_{k+i} = \frac{(-1)^{k+i}}{2\pi\sqrt{\lambda_x\lambda_y}\;k!i!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\xi_x^2}{2\lambda_x} + \frac{\xi_y^2}{2\lambda_y}\right)^{k+i}$$

$$\cdot\ \exp\left|j(\xi_x\hat{\xi}_x + \xi_y\hat{\xi}_y) - \frac{1}{2}\,(\xi_x^2 + \xi_y^2)\right|\ d\xi_x d\xi_y. \quad \text{(D-12)}$$

Using the Binomial Expansion Theorem in equation (D-12) yields

$$\Omega_{k+i} = \frac{(-1)^{k+i}}{2\pi\sqrt{\lambda_x\lambda_y}\,k!\,i!} \sum_{\ell=o}^{k+i} \left[\frac{(k+i)!}{(k+i-\ell)!\,\ell!}\right] \int_{-\infty}^{\infty} \left(\frac{\xi_x^2}{2\lambda_x}\right)^{k+i-\ell} e^{j\xi_x\hat{\xi}_x - \frac{1}{2}\xi_x^2}\, d\xi_x$$

$$\cdot \int_{-\infty}^{\infty} \left(\frac{\xi_y^2}{2\lambda_y}\right)^{\ell} e^{j\xi_y\hat{\xi}_y - \frac{1}{2}\xi_y^2}\, d\xi_y \quad . \tag{D-13}$$

Equation (D-13) can be written as

$$\Omega_{k+i} = \frac{(-1)^{k+i}}{2\pi\sqrt{\lambda_x\lambda_y}\;k!\,i!} \sum_{\ell=o}^{k+i} \left[\frac{(k+i)!}{(k+i-\ell)!\,\ell!}\; f_{k+i-\ell}(\xi_x,\lambda_x) f_\ell(\xi_y,\lambda_y)\right] \tag{D-14a}$$

where

$$f_{k+i-\ell}(\hat{\xi}_x,\lambda_x) = \int_{-\infty}^{\infty} \left(\frac{\xi_x^2}{2\lambda_y}\right)^{k+i-\ell} e^{j\xi_x\hat{\xi}_x - \frac{1}{2}\xi_x^2}\; d\xi_x \tag{D-14b}$$

and

$$f_\ell(\hat{\xi}_y, \lambda_y) = \int_{-\infty}^{\infty} \left(\frac{\xi_x^2}{2\lambda_y}\right)^{\ell} e^{j\xi_y\hat{\xi}_y - \frac{1}{2}\xi_y^2}\; d\xi_y . \tag{D-14c}$$

Making the substitution, $\xi_x = \sqrt{2Z}$ in equation (D-14b) yields

$$f_m(\hat{\xi}_x, \lambda_x) = \frac{\sqrt{2}}{\lambda_x^{\,m}} \int_{\infty}^{\infty} Z^{2m} \exp\left\{ j2Z \frac{\hat{\xi}_x}{\sqrt{2}} - Z^2 \right\} dZ \qquad \text{(D-15)}$$

where $m = k + i - \ell$.

Equation (D-15) can be integrated to yield (from reference 7)

$$f_m(\hat{\xi}_x,\lambda_x) = \frac{\sqrt{2\pi}}{\lambda_x^m} (-1)^m \exp\left\{ \frac{-\hat{\xi}_x}{2} \right\} 2^{-2m} H_{2m}\left(\frac{\hat{\xi}_x}{\sqrt{2}}\right) \qquad \text{(D-16a)}$$

where $H_{2m}\left(\frac{\hat{\xi}_x}{\sqrt{2}}\right)$ is a Hermite polynomial given by

$$H_{2m}\left(\frac{\hat{\xi}_x}{\sqrt{2}}\right) = \sum_{k=o}^{m} \frac{(-1)^k 2^{m-k}(2m)!}{k! \quad (2m-2k)!} \left[\hat{\xi}_x^2\right]^{m-k}. \qquad \text{(D-16b)}$$

Substituting equation (D-16b) into equation (D-16a) and reversing the summation yields

$$f_m(\hat{\xi}_x, \lambda_x) = \sqrt{2\pi} \exp\left\{ - \frac{\hat{\xi}_x^2}{2} \right\} 2^{-2m}(2m)! \sum_{k=o}^{m} \frac{2^k \left(-\hat{\xi}_x^2\right)^k}{(m-k)!(2k)!}. \qquad \text{(D-17)}$$

The factorials (2k)! and (2m)! can be written as

$$(2A)! = \left[2A\ (2A-2)\ (2A-4)\ (2A-6) \ldots\ldots 6.4.2\right]$$

$$\cdot \left[(2A-1)\ (2A-3)\ (2A-5) \ldots\ldots 5.3.1\right]$$

which reduces to

$$(2A)! = 2^A\ A!\ (2A-1)!! \tag{D-18}$$

where $(2A-1)!! = (2A-1)\ (2A-3) \ldots\ldots 5.3.1.$

Substituting equation (D-18) for (2k)! and (2m)!, equation (D-17) becomes

$$f_m(\hat{\xi}_x, \lambda_x) = \sqrt{2\pi}\ m!\ \exp\left\{\frac{-\hat{\xi}_x^2}{2}\right\}\left[\frac{1}{4\lambda_x}\right]^m$$

$$\cdot \binom{2m}{m} \sum_{k=o}^{m} \binom{m}{k} \frac{\left(-\hat{\xi}_x^2\right)^k}{(2k-1)!!}\ . \tag{D-19}$$

Let

$$\Theta_m(\hat{\xi}_x, \lambda_x) = \left[-\frac{1}{4\lambda_x}\right]^m \binom{2m}{m} \sum_{k=o}^{m} \binom{m}{k} \frac{\left(-\hat{\xi}_x^2\right)^k}{(2k-1)!!}$$

then

$$f_m(\hat{\xi}_x, \lambda_x) = \sqrt{2\pi}\, m!(-1)^m \exp\left\{-\frac{\hat{\xi}_x^2}{2}\right\} \Theta_m(\hat{\xi}_x, \lambda_x) \; . \qquad \text{(D-20)}$$

Solving equation (D-14c) for $f_\ell(\hat{\xi}_y,\lambda_y)$ results in a similar equation of the form

$$f_\ell(\hat{\xi}_y,\lambda_y) = \sqrt{2\pi}\; \ell!(-1)^\ell \exp\left\{\frac{-\hat{\xi}_y^2}{2}\right\} \Theta_\ell\left(\hat{\xi}_y, \lambda_y\right). \qquad \text{(D-21)}$$

Substituting equations (D-20) and (D-21) into equation (D-14a) yields

$$\Omega_{k+i} = \frac{1}{\sqrt{\lambda_x \lambda_y}} \binom{k+i}{k} \sum_{\ell=o}^{k+i} e^{-\frac{1}{2}(\hat{\xi}_x^2 + \hat{\xi}_y^2)}\, \Theta_{k+i-\ell}(\hat{\xi}_x,\lambda_x)\Theta_\ell(\hat{\xi}_y,\lambda_y). \qquad \text{(D-22)}$$

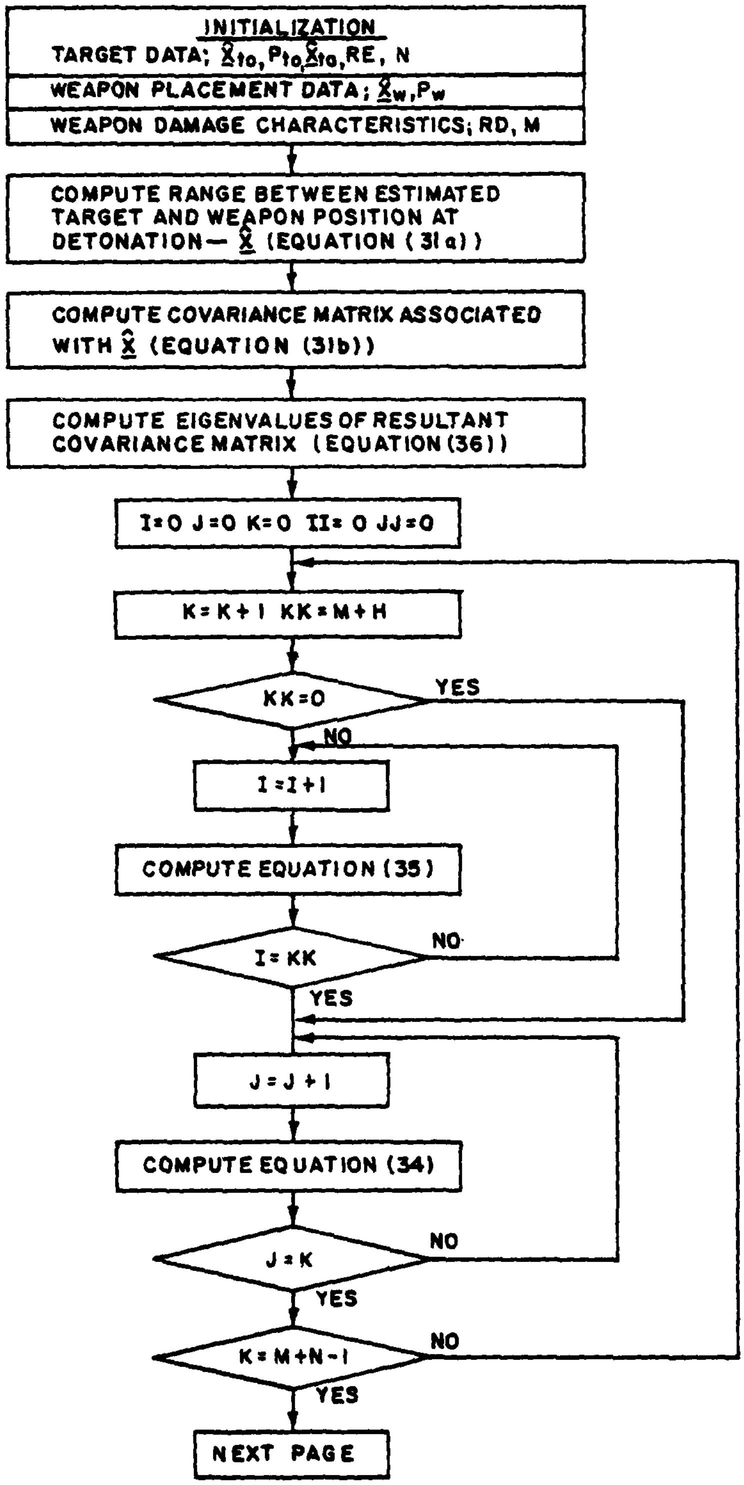
APPENDIX E

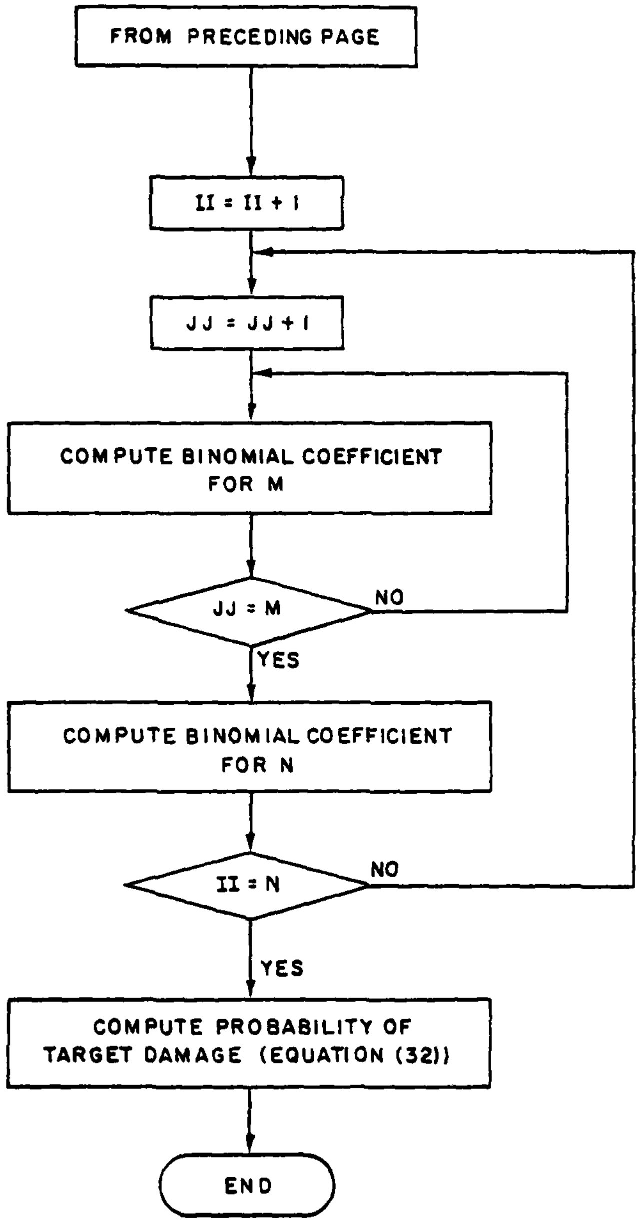
FORTRAN PROGRAM (WITH FLOWCHART)

TO EVALUATE THE PROBABILITY OF TARGET DAMAGE

```
&L
      SUBROUTINE MISSILE(PROB)
      COMMON /INPUT/XTO(2),PTO(2,2),XTA(2),RE,N,XW(2),PW(2,2),
     1RD,M
      DIMENSION THETAX(19),THETAY(19),AREA(19)
      IF(M.LT.1) M=1
      IF(N.LT.1) N=1
      IF(M.GT.10) M=10
      IF(N.GT.10) N=10
      Z = 2*(N+1)
      RHOE = (RE*RE)/Z
      Z = 2*M
      RHOD = (RD*RD)/Z
      ALPHAX = 0,5*(PTO(1,1)+PW(1,1)+RHOE+RHOD)
      ALPHAY = 0,5*(PTO(2,2)+PW(2,2)+RHOE+RHOD)
      Z = PTO(1,2)+PW(1,2)
      ZZ = ALPHAX+ALPHAY
      ZZZ = ALPHAX-ALPHAY
      ALPHAX = SQRT(ZZZ*ZZZ+Z*Z)
      EIGENX = 1.0/(ZZ+ALPHAX)
      EIGENY = 1.0/(ZZ-ALPHAX)
      ZZ = ALPHAX-ZZZ
      Z = ATAN2(Z/ZZ)
      ALPHAX = XTO(1)+XTA(1)-XW(1)
      ALPHAY = XTO(2)+XTA(2)-XW(2)
      ETAX = ALPHAX*SIN(Z)+ALPHAY*COS(Z)
      ETAY = ALPHAY*SIN(Z)-ALPHAX*COS(Z)
      ETAX = -ETAX*ETAX*EIGENX
      ETAY = -ETAY*ETAY*EIGENY
      Z = N
      ZZ = 1.0/(Z*RHOE)
      PROB = ZZ*SQRT(EIGENX*EIGENY)*EXP(0.5*(ETAX+ETAY))
      EIGENX = -0.0*EIGENX
      EIGENY = -0.0*EIGENY
      MM = M+N-1
      ALPHAX = 1.0
      ALPHAY = 1.0
      DO 40 K = 1, MM
      THETAX(K) = 1.0
      THETAY(K) = 1.0
      KK = K-1
```

```
      IF(KK.EQ.0) GO TO 20
      Z = 1.0
      DO 10 I =1,KK
      ZZ = I
      ZZZ = K-I
      Z = (ZZZ/ZZ)*Z
      THETAX(K) = Z+((ETAX*THETAX(K))/(2.0*ZZZ-1.0))
      THETAY(K) = Z+((ETAY*THETAY(K))/(2.0*ZZZ-1.0))
   10 CONTINUE
      ZZ = (2.0*ZZ-1.0)/ZZ
      ALPHAX = EIGENX*ZZ*ALPHAX
      ALPHAY = EIGENY*ZZ*ALPHAY
      THETAX(K) = ALPHAX*THETAX(K)
      THETAY(K) = ALPHAY*THETAY(K)
   20 CONTINUE
      OMEGA(K) = 0.0
      DO 30 I = 1,K
      KK = K+1-I
      OMEGA(K) = OMEGA(K)+THETAX(I)*THETAY(KK)
   30 CONTINUE
   40 CONTINUE
      ZZ = 1.0
      ZZZ = 0.0
      DO 60 K = 1,N
      Z = 0.0
      DO 50 I = 1,N
      KK = M+K-I
      ALPHAX = I*(KK-1+(I/N))
      ALPHAY = (N-I+1)*(N-I+(I/N)*K)
      Z = (ALPHAX*RHOE*(Z+OMEGA(K)))/ALPHAY
   50 CONTINUE
      ALPHAX = N-K+I
      ALPHAY = K
      ZZ = (ALPHAX*RHOE*ZZ)/ALPHAY
      ZZZ = ZZZ+ZZ*Z
   60 CONTINUE
      PROB = PROB*ZZZ
      RETURN
      END
*
```


INITIALIZATION
TARGET DATA; $\hat{X}_{to}$, P_{to}, $\hat{X}_{to}$, RE, N
WEAPON PLACEMENT DATA; $\hat{X}_w$, P_w
WEAPON DAMAGE CHARACTERISTICS; RD, M
COMPUTE RANGE BETWEEN ESTIMATED TARGET AND WEAPON POSITION AT DETONATION — $\hat{X}$ (EQUATION (31a))
COMPUTE COVARIANCE MATRIX ASSOCIATED WITH $\hat{X}$ (EQUATION (31b))
COMPUTE EIGENVALUES OF RESULTANT COVARIANCE MATRIX (EQUATION (36))
I=0 J=0 K=0 II=0 JJ=0
K=K+1 KK=M+H
KK=0
YES
NO
I=I+1
COMPUTE EQUATION (35)
I=KK
NO
YES
J=J+1
COMPUTE EQUATION (34)
J=K
NO
YES
K=M+N-1
NO
YES
NEXT PAGE


FROM PRECEDING PAGE
II = II + 1
JJ = JJ + 1
COMPUTE BINOMIAL COEFFICIENT FOR M
JJ = M
NO
YES
COMPUTE BINOMIAL COEFFICIENT FOR N
II = N
NO
YES
COMPUTE PROBABILITY OF TARGET DAMAGE (EQUATION (32))
END

REFERENCES

1. A. Papoulis, Probability, Random Values, and Stochastic Processes, McGraw-Hill Book Co., New York, 1975.

2. N. N. Lebedev, Special Functions and Their Applications, Prentice-Hall, Inc., New Jersey, 1965.

3. W. B. Davenport and W. L. Root, An Introduction to the Theory of Random Signals and Noise, McGraw-Hill Book Co., New York, 1958.

A-1. A. W. Goodman, Modern Calculus with Analytical Geometry, The MacMillan Company, New York, 1968.

A-2. I. S. Gradshteyn and I. M. Ryzhik, Tables of Integrals Series and Products, Academic Press, New York, 1965.

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